B.Sc. (Hon's), Ist Semester Examination, 2013
Mathematics (Algebra)
AS - 2819

Q No. 1.

(i) Rank of the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, i.e. $\rho(A) = 2$.

(ii) If every minor of order r of matrix A is zero, then the rank of the matrix A, $\rho(A) < r$.

(iii) Since the system of homogeneous linear equations has Infinite solutions, So the determinant of Coefficients matrix is equal to zero.

i.e. $\begin{vmatrix} 1 & 2 & 3 \\ 1 & t & 1 \\ 1 & 3 & 1 \end{vmatrix} = 0 \Rightarrow 1(t-3) - 2(1-1) + 3(3-t) = 0$
$\Rightarrow 6 - 2t = 0$
$\Rightarrow t = 3$

(iv) Let e be the identity of the group of Integers **I**.

Given, $a o b = a + b + 1$ $\quad a, b \in I$

Now, $a o e = a$ ——— (i)

and $a o e = a + e + 1$ ——— (ii)

From (i) and (ii), We get $a + e + 1 = a \Rightarrow e = -1$.

(v) Centre of a group :- Let G be a group. Then

$Z(G) = \{ x \in G : xg = gx \ \forall g \in G \}$ is Called Centre of a group G.

(vi) Quaternion group is a non-abelian group whose all subgroups are normal.

$Q_8 = \{ \pm 1, \pm i, \pm j, \pm k \}$. Define product by usual multiplication together with

$i^2 = j^2 = k^2 = -1$, $\quad ij = -ji = k$
$jk = -kj = i$
$ki = -ik = j$.

(vii) Kernel of homomorphism of a group G :- Let $f : G \to G'$ is a homomorphism, Then kernel of f, denoted by Kerf is given by

$\text{Ker} f = \{ x \in G : f(x) = e'$, where e' is the identity of $G' \}$.

(viii) $8 = 2 \times 2 \times 2 = 2^3$

No. of generators $= \phi(8)$
$= 8 \cdot (1 - 1/2)$
$= \cancel{8} \cdot 1/\cancel{2}$
$= 4$

Therefore, 4 elements can be used as generators of the group.

Q No. 2. (a)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since in echelon form of a matrix, There are only two non-zero rows. Hence rank of the matrix A, i.e. $\rho(A) = 2$. Ans.

(b): We know that $A = IA$

$$\therefore \begin{bmatrix} 0 & 2 & 3 \\ 1 & 6 & 4 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boxed{A}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} A$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -2 \end{bmatrix} A$$

$R_1 \to R_1 + 4R_3$, $R_2 \to R_2 + 2R_3$

$$\begin{bmatrix} 1 & 6 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 1 & -8 \\ 2 & 0 & -3 \\ 1 & 0 & -2 \end{bmatrix} A$$

$R_1 \to R_1 - 6R_2$, $R_3 \to -1R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -8 & 1 & 10 \\ 2 & 0 & -3 \\ -1 & 0 & 2 \end{bmatrix} A$$

Hence $A^{-1} = \begin{bmatrix} -8 & 1 & 10 \\ 2 & 0 & -3 \\ -1 & 0 & 2 \end{bmatrix}$. Ans.

Q No. 3.

$x - y + 2z = 5$

$3x + y + z = 8$

$2x - 2y + 3z = 7$

Augmented matrix $C = [A : B]$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{ccc:c} 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \\ 2 & -2 & 3 & 7 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & -1 & 2 & 5 \\ 0 & 4 & -5 & -7 \\ 0 & 0 & -1 & -3 \end{array}\right]$$

Since the rank of A = Rank of C = 3 (No. of Unknown Variables)

Hence the system of linear equations is consistent.

Now, $x - y + 2z = 5$ —— <i>

$4y - 5z = -7$ —— <ii>

$\cancel{-}z = \cancel{-}3 \Rightarrow z = 3$

put $z = 3$ in <ii> We get,

$4y - 5 \times 3 = -7$

$4y = -7 + 15 = 8 \Rightarrow y = \cancel{8}^{2}/\cancel{4} = 2$

Putting $z = 3$, $y = 2$ in (i)

We get,

$x - 2 + 2 \times 3 = 5$

$\Rightarrow x = 5 - 4 = 1$

Hence, $x = 1$

$y = 2$ Ans.

$z = 3$

Q.No. 4. Let $S = \{ m+n\sqrt{2}$, where $m, n \in Q \}$
We have to show that $(S, \cdot)$ is a group.

Closure:- Let $m_1+n_1\sqrt{2}$, $m_2+n_2\sqrt{2} \in S$
where $m_1, m_2, n_1, n_2 \in Q$

Now, $(m_1+n_1\sqrt{2})\cdot(m_2+n_2\sqrt{2}) = m_1m_2 + m_1n_2\sqrt{2} + n_1m_2\sqrt{2} + 2n_1n_2$
$= (m_1m_2 + 2n_1n_2) + (m_1n_2 + n_1m_2)\sqrt{2} \in S$

as $(m_1m_2+2n_1n_2), (m_1n_2+n_1m_2) \in Q$.
Hence Closure axiom is satisfied.

Associative:- Let $m_1+n_1\sqrt{2}$, $m_2+n_2\sqrt{2}$, $m_3+n_3\sqrt{2} \in S$

$$[(m_1+n_1\sqrt{2})\cdot(m_2+n_2\sqrt{2})](m_3+n_3\sqrt{2})$$
$$= (m_1m_2m_3+2n_1n_2m_3+2m_1n_2m_3+2n_1m_2m_3) + (m_1m_2n_3+2n_1n_2n_3+m_1m_3n_2+n_1m_2m_3)\sqrt{2}$$

and $(m_1+n_1\sqrt{2})\cdot[(m_2+n_2\sqrt{2})\cdot(m_3+n_3\sqrt{2})]$
$$= (m_1m_2m_3+2n_1n_2m_3+2m_1n_2m_3+2n_1m_2m_3) + (m_1m_2n_3+2n_1n_2n_3+m_1m_3n_2+n_1m_2m_3)\sqrt{2}$$

i.e. $[(m_1+n_1\sqrt{2})\cdot(m_2+n_2\sqrt{2})](m_3+n_3\sqrt{2}) = (m_1+n_1\sqrt{2})\cdot[(m_2+n_2\sqrt{2})\cdot(m_3+n_3\sqrt{2})]$

Hence associative axiom is satisfied.

Identity:- Let $m+n\sqrt{2} \in S$ and e be the identity of S. then
$(m+n\sqrt{2})\cdot e = (m+n\sqrt{2}) \Rightarrow e = \frac{m+n\sqrt{2}}{m+n\sqrt{2}} = 1$

i.e. $1 = (1+0.\sqrt{2}) \in S$ is an identity element w.r.t. multiplication.

Inverse:- let $(m+n\sqrt{2}) \in S$, then
$$(m+n\sqrt{2})^{-1} = \frac{1}{m+n\sqrt{2}} = \frac{m-n\sqrt{2}}{m^2-2n^2} = \left(\frac{m}{m^2-2n^2}\right) + \left(\frac{-n}{m^2-2n^2}\right)\sqrt{2}$$
$$= \alpha + \beta\sqrt{2} \in S$$

where $\alpha = \frac{m}{m^2-2n^2}$, $\beta = \frac{-n}{m^2-2n^2} \in Q$, $\forall\, m, n \in Q$

Hence the inverse axiom is satisfied.

Therefore, $(S, \cdot)$ is a group.

Q No. 5 (a) Let $o(G) = n$

Since Corresponding to each element in G, We Can define a right Coset of H in G. The number of distinct Cosets of H in G is less than or equal to n.

We have, $G = Ha_1 \cup Ha_2 \cup \cdots\cdots \cup Ha_t$

where $t =$ No. of distinct right Cosets of H in G.

$\Rightarrow o(G) = o(Ha_1) + o(Ha_2) + \cdots\cdots + o(Ha_t)$.

$\Rightarrow o(G) = o(H) + o(H) + \cdots\cdots + o(H)$ (t-times)

$\Rightarrow o(G) = t \cdot o(H)$

$\Rightarrow o(H) / o(G)$ but Converse does not hold. Proved.

It is Called Lagrange's Theorem.

(b) Let $Ha = Hb$

$\Rightarrow (Ha)b^{-1} = (Hb)b^{-1}$

$\Rightarrow Hab^{-1} = Hbb^{-1}$

$\Rightarrow Hab^{-1} = He$

$\Rightarrow Hab^{-1} = H$

$\Rightarrow ab^{-1} \in H$

Conversely, let $ab^{-1} \in H$

$\Rightarrow Hab^{-1} = H$

$\Rightarrow (Hab^{-1})b = Hb$

$\Rightarrow Hab^{-1}b = Hb$

$\Rightarrow Ha.e = Hb$

$\Rightarrow Ha = Hb$ Proved.

Q No. 6 (a) Let H be a Subgroup of G with index 2 then number of distinct right (left) Coset of H in G is 2 and also then G is union of these two right (left) Cosets.

let $g \in G$ be arbitrary element.

Case (i) : $g \in H$, Then $Hg = gH$ $(= H)$, Hence H is normal.

Case (ii) : $g \notin H$, then $gH \neq H$, $Hg \neq H$

Thus Hg and $H = He$ are the two distinct right Cosets of H in G.

Now, $G = Hg \cup H$

Similarly, $G = gH \cup H$

$\Rightarrow Hg \cup H = gH \cup H$

$\Rightarrow Hg = gH$ (as $Hg \cap H = gH \cap H = \phi$)

$\Rightarrow H$ is normal in G.

6.(b). Let $h \in H$ $k \in K$ be any elements.

Then $h \in H$, $k \in K \subseteq G$, H is normal in G

gives $k^{-1} h k \in H \Rightarrow k^{-1} h k h^{-1} \in H$

Again $h^{-1} \in H \subseteq G$, $k \in K$, K is normal in G

gives $(h^{-1})^{-1} k h^{-1} \in K \Rightarrow h k h^{-1} \in K \Rightarrow k^{-1} h k h^{-1} \in K$

i.e. $k^{-1} h k h^{-1} \in H \cap K = \{e\}$

$\Rightarrow k^{-1} h k h^{-1} = e \Rightarrow hk = kh$ Proved.

Q No. 7. statement of Fundamental Theorem of group homomorphism :-

If $f: G \to G'$ be an onto homomorphism with $K = \operatorname{Ker} f$, then $\frac{G}{K} \cong G'$.

In other words, every homomorphic image of a group G is Isomorphic to a quotient group of G.

Proof:- Define a map $\phi: \frac{G}{K} \to G'$ s.t. $\phi(Ka) = f(a)$, $a \in G$.

ϕ is well defined as $Ka = Kb \Rightarrow ab^{-1} \in K = \operatorname{Ker} f \Rightarrow f(ab^{-1}) = e'$

$\Rightarrow f(a)\,(f(b))^{-1} = e' \Rightarrow f(a) = f(b) \Rightarrow \phi(Ka) = \phi(Kb)$.

ϕ is one-one follows by

$\phi(Ka) = \phi(Kb)$

$\Rightarrow f(a) = f(b) \Rightarrow f(a)\,[f(b)]^{-1} = f(b)\,[f(b)]^{-1} \Rightarrow f(a)\, f(b^{-1}) = e'$

$\Rightarrow f(ab^{-1}) = e' \Rightarrow ab^{-1} \in K \Rightarrow Ka = Kb$.

ϕ is onto as, let $g' \in G'$ be any element. Since $f: G \to G'$ is onto,

$\exists\, g \in G$ s.t. $f(g) = g'$

Now $\phi(Kg) = f(g) = g'$ i.e Kg is the required pre-image of g' under ϕ.

Now $\phi[Ka \cdot Kb] = \phi(Kab) = f(ab) = f(a)f(b) = \phi(Ka)\phi(Kb)$

i.e. ϕ is a homomorphism.

Thus we have that ϕ is a one-one onto homomorphism and hence ϕ is an Isomorphism. i.e. $\frac{G}{K} \cong G'$.

QNo. 8 (a). Let $G = \langle a \rangle$ be of order n.

Define $f: Z \to G$ s.t. $f(m) = a^m$

Then f is clearly well defined one-one onto map.

Since $f(m+k) = a^{m+k} = a^m \cdot a^k = f(m) \cdot f(k)$, f is also a homomorphism and

Therefore, by Fundamental theorem $G \cong \frac{Z}{\mathrm{Ker} f}$

Now we have to show $\mathrm{Ker} f = N = \langle n \rangle$.

Let $m \in \mathrm{Ker} f \Leftrightarrow f(m) = e$, e be the identity of G.

$\Leftrightarrow a^m = e$

$\Leftrightarrow o(a)/m$

$\Leftrightarrow n/m$

$\Leftrightarrow m \in \langle n \rangle = N$

$\Rightarrow \mathrm{ker} f = N$

Hence $G \cong \frac{Z}{N} = \frac{Z}{\langle n \rangle}$ Proved.

(b). Let the order of a generator 'a' of a cyclic group be n, i.e $o(a) = n$ Then $a^n = e$ while $a^s \neq e$ for $0 < s < n$

When $s > n$, $s = nq + r$, $0 \leq r < n$

Now $a^s = a^{nq+r} = (a^n)^q \cdot a^r = e^q \cdot a^r = e \cdot a^r = a^r$

Thus there are exactly n elements in the group given by a^r where $0 \leq r < n$.
Therefore, there are n and only n distinct elements in the cyclic group, i.e. the order of the group is n. Proved.

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